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ON THE STABILITY CONDITIONS FOR STATIONARY STATES OR FLOWS IN REGIONS EXTENDED IN ONE DIRECTION*

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A generalization of the asymptotic condition of stability (/1/) for stationary states or flows in the segment $0 \leqslant x \leqslant L$ for large L is given. Unlike /1/, the existence inside this segment of points where the boundary conditions are also specified, are allowed. No constraints are imposed on the boundary conditions with the exception of those resulting from the requirement of accuracy. Unlike the case in /1/, these boundary conditions can be degenerate, that is the vanishing of any number of reflection coefficients or refractive indices of various disturbances are admissible. In addition, a state or a flow can slowly change when x changes, i.e. it can depend on x/L. In this case the distrubances may be reflected not only from points where the boundary conditions are specified, but also they can be affected by internal reflections from reflection points, or the points of intersection of the real x-axis with the Stokes lines (see, for example, /2/) in the complex plane.

It is shown that if we exclude the instability created by the boundary conditions set at one of the points ('boundary' instability), then, in general, the instability condition would be the existence, for $\operatorname{Im}\omega > 0$, of a cyclic sequence of waves which depend on time as $e^{-i\omega t}$, and which are converted from one into another, the sequence being such that the product of the space amplification (or attenuation) coefficients of these waves, and their mutual conversion coefficient on reflection or refraction should be unity. As applied to weak non-uniform states and flows, the above condition can be regarded as an extension to the arbitrary boundary conditions of the 'quantization conditions' obtained when there are only internal reflections from the Stokes lines, or from the points of rotation (see /2-7/).

1. Consider the behaviour of the distrubances of an arbitrary state of flow which depend on $x^* = x/L$ in the section $0 \leqslant x \leqslant L$ for large L. We assume that at the point $x = X^\alpha$ ($\alpha = 0, 1, \ldots, \mu+1$), $X^0 = 0$, $X^\mu = L$ distant from each other a distance of order L, are the boundary conditions which connect the disturbances and their derivatives at these points are specified.

We shall assume that in each of the segments $[X^\alpha, X^{\alpha+1}]$ the disturbances are described by a linear system of equations for which we assume the following. Any solution of this system $u_j^\alpha(x, t)$ on the segment $[X^\alpha, X^{\alpha+1}]$, which depends on time as $e^{-i\omega t}$, is presented in the form of a linear combination of the independent solutions

$$u_j^\alpha(\omega, x) = \sum_m C_m^\alpha u_{mj}^\alpha(\omega, x^*)(1 - R_{mj}^\alpha) \exp\left(i \int k_m^\alpha(\omega, x^*)\, dx\right) \tag{1.1}$$

$$(R_{mj}^\alpha = R_{mj}^\alpha(\omega, x^*, L) \to 0,\ L \to \infty)$$

The function $u_{mj}^\alpha(\omega, x^*)$ satisfies the condition $\left(\sum_j |u_{mj}^\alpha|^2\right)^{1/2} \equiv u_m^\alpha \neq 0$. Here $k_m^\alpha(\omega, x^*)$ are the roots of the dispersion equation

$$\Phi_\alpha(\omega, k, x^*) = 0 \tag{1.2}$$

in which the slow variable x^* appears as a parameter. We will assume that the values of k^α which satisfy this equation, form the finite set $k_m^\alpha(\omega, x^*)$ ($m = 1, 2, \ldots, N^\alpha$), or for some other reasons, we can limit ourselves in the sum (1.1) to N^α terms only. Eq.(1.1) corresponds to the WKB approximation (/2/).

Suppose that the condition of correctness (see /8/) (or of "being evolutionary") of a system of equations is satisfied. This condition stipulates the existence of a constant M such that for $\operatorname{Im}\omega > M$ for all x^* and α the imaginary parts of all k_m^α are non-zero. This means that for $\operatorname{Im}\omega > M$ the change of the sign of $\operatorname{Im} k_m^\alpha$ can occur only when k_m^α passes through infinity. Let us assume that such passage can occur only at certain isolated points

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x, and at those points we can stipulate certain effective boundary conditions which link the solutions situated in different directions from these points. Then, assuming that these points are included among the points X^α we obtain that for $\operatorname{Im}\omega > M$, the quantities $\operatorname{Im} k_l^\alpha$ do not change sign on any of the segments $[X^\alpha, X^{\alpha-1}]$.

Let us introduce numbers s^α such that for $\operatorname{Im}\omega > M$,

$$\begin{aligned} &\operatorname{Im} k_j^\alpha(\omega, x^*) > 0 \ (j = 1, 2, \ldots, s^\alpha) \\ &\operatorname{Im} k_p^\alpha(\omega, x^*) < 0 \ (p = s^{\alpha+1}, \ldots, N^\alpha) \end{aligned} \tag{1.3}$$

We assume that the boundary conditions for disturbances are independent of L, they are uniform and are separating (i.e. they link the values of the disturbances and their derivatives at the points where the conditions are specified), and they satisfy the necessary conditions of correctness (see /9, 10/). The conditions can be expressed as follows. For $x = 0$ and $x = L$ we must set s^0 and $N^\mu - s^\mu$ boundary conditions respectively, and at each of the inner points X^α there should be $N^{\alpha-1} - s^{\alpha-1} + s^\alpha$ boundary conditions not counting the equation $x = X^\alpha$ which gives the position of the point itself. Generally speaking, the quantities X^α may become functions of frequency ω.

After substituting the solutions of (1.1) into the boundary conditions the latter become a homogeneous linear system of algebraic equations with coefficients, depending on ω, regarding C_m^α. For $x = 0$ and $x = L$, the boundary conditions connect the values of C_m^0 and of C_m^μ between themselves respectively. Figure 1 shows the form of the matrix Δ of all coefficients of C_m^λ.

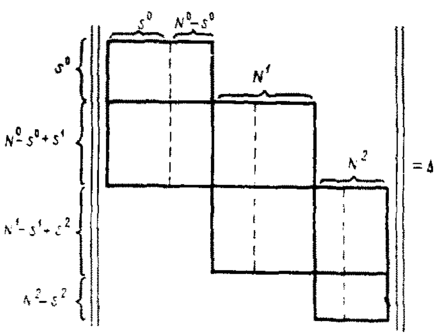

Fig.1

For simplicity, we show the case of $\mu = 2$, that is when the segment $[0, L]$ is divided into three parts by the points X^1 and X^2. Outside the rectangles shown by the solid lines, all elements are zeros. The symbols on the left and the top indicate the number of rows and columns in the corresponding minors.

Equating the determinant of the matrix Δ to zero, we obtain an equation for finding the natural frequencies ω.

Each term of (1.1) can be regarded as a wave to which we can assign a direction of propagation depending on the sign of $\operatorname{Im} k^\alpha_m$, which this quantity assumes for $\operatorname{Im}\omega > M$ (see /1/). The waves which correspond to $k_1^\alpha, k_2^\alpha, \ldots, k_{s^\alpha}^\alpha$ $(k_{s^{\alpha+1}}^\alpha, \ldots, k_{N^\alpha}^\alpha)$. are considered as propagating to to the right or left. For $\operatorname{Im}\omega > M$, all waves suffer space attenuation in their direction of propagation. At each point where they are stipulated, the boundary conditions should enable us to distinguish departing waves from arriving ones. The number of boundary conditions necessary for correctness depends on this. The vertical dashed lines in Fig.1 separate the columns of Δ which correspond to C_m^α regarding waves propagating in opposite directions.

Let us introduce the notation

$$iK_m^\alpha = \pm\frac{1}{L}\int\limits_{X^\alpha}^{X^{\alpha-1}}\left[ik_m^\alpha(\omega, x^*) - \frac{1}{u_m^\alpha}\frac{\partial u_m^\alpha}{\partial x}\right]dx \tag{1.4}$$

(the plus sign is taken for $m = 1, 2, \ldots, s^\alpha$, and the minus sign for $m = s^{\alpha+1}, \ldots, N^\alpha$). Clearly, K_m^α depends on ω, and also on X^α and $X^{\alpha-1}$. Supposing that the latter quantities are functions of ω and L, and $X^{\alpha-1} - X^\alpha$ is of order L, we shall assume that the limit of expression (1.4)

$$\lim_{L\to\infty} K_m^\alpha = K_m^{*\alpha}(\omega)$$

exists.

If in the indefinite integral in the exponent of (1.1) we choose the integration constant so that at the end of the segment $[X^\alpha, X^{\alpha-1}]$ from which the corresponding wave emerges, the modulus of this wave is $|C_m^\alpha|$, then at the other end of this segment the wave modulus is $|C_m^\alpha|\exp(-\operatorname{Im} K_m^\alpha L)$. The waves themselves at the respective ends of the segment are given as follows:

$$C_m^\alpha \frac{u_{mj}^\alpha}{u_m^\alpha}, \qquad C_m^\alpha \frac{u_{mj}^\alpha}{u_m^\alpha}\exp iLK_m^\alpha \tag{1.5}$$

We shall describe $-\operatorname{Im} LK_m^\alpha$ the exponents of space wave amplification. Obviously, in expression (1.4) the first term is the main term which, as $L \to \infty$, remains finite, and the contribution of the second term to K_m^α tends to zero. In accordance with Eqs.(1.2), for $\operatorname{Im}\omega > M$ all exponents of the space amplification of waves are negative.

Because the boundary conditions are independent of L, all elements of the matrix Δ, which correspond to waves departing from a point where the boundary condition is stipulated, will be of order one, and the elements in the column corresponding to the coefficient C_m^α for the arriving wave will contain the factor $\exp iLK_m^\alpha$. For $\operatorname{Im}\omega > M$ and $L \to \infty$, all these elements tend to zero so that in the limit only the elements of the square minors remain, which lie on the principle diagonal and in Fig.1 are bounded by the dashed lines.

This means that when $\operatorname{Im}\omega > M$, for sufficiently large L the determinant of the matrix Δ is approximately equal to the product of the determinants of the above-mentioned minors,

$$|\Delta(\omega)| = |A_0(\omega)|\,|A_1(\omega)|\ldots|A_\mu(\omega)| \tag{1.6}$$

As long as the expression written in (1.6) remains greater in order of magnitude than that of the other terms, the fact that the determinant of the matrix Δ equals zero as $L \to \infty$ results in the vanishing of at least one determinant, for example

$$|A_\nu(\omega)| = 0 \tag{1.7}$$

The system of boundary conditions which corresponds to this determinant links the values of the disturbance and their derivative between themselves at the point $x = X^\nu$ whose situation is a function of ω. Therefore the left-hand side of Eq.(1.7) does not contain a dependence on L. In the complex plane ω, the separate points whose situation does not depend on L can correspond to the roots of Eq.(1.7). There may be no such points if all $|A_\nu|$ are constant.

In satisfying Eq.(1.7) we can construct a solution in which C_m^ν $(m = 1, \ldots, s^\nu)$ and $C_j^{\nu+1}$ $(j = s^{\nu+1}, \ldots, N^{\nu+1})$ are different from zero, and correspond to the waves departing from the point X^ν. As $L \to \infty$, the remaining C_m^β for $\operatorname{Im}\omega > M$ equal zero in the limit. For $\operatorname{Im}\omega < M$, some other C_m^β may differ from zero as well; however the corresponding waves do not exert any influence on the development of turbulence or on the value of ω as long as Eq.(1.6) holds.

If Eq.(1.7) holds for $\operatorname{Im}\omega > 0$, an instability occurs connected with the increase in disturbances generated by the point X^ν. As is evident from (1.7), the conditions for such an instability to appear are connected with specifying the boundary conditions at this point. We shall call it a boundary instability.

The other form of the eigenfunctions and the instability appears for $\omega < M$ when the disregarded terms in (1.6) become comparable with the expression on the right-hand side of this equation.

Consider the case where the determinants of all the minors A_ν are different from zero, disregarding in the complex plane of ω small neighbourhoods of the points at which this condition is not satisfied. Then the boundary condition can be resolved with respect to the amplitudes of the departing waves. Here all minors A_ν will become unit matrices, and outside these minors in the same rows will be elements each of which contains a factor $\exp iLK_m^\alpha$ determined by the number of a column. The factor of this exponent is, with the opposite sign, a conversion coefficient (i.e. the reflection coefficient or refractive index) of the incident and the departing wave.

The determinant of the matrix Δ can be calculated if we present it as a sum which corresponds to different X^ν, of the minors' determinants compiled from the columns appearing in each system of rows. Each such product contains the exponential factor $\exp i\sum LK_m^\alpha$, where the summation is taken over the numbers α and m of the columns, which occur in one of the minors indicated, and do not appear in the minors A_α. Thus we can write symbolically

$$|\Delta| = 1 + \sum(\ldots)(\ldots)\ldots(\ldots)\exp i\sum LK_m^\alpha \tag{1.8}$$

where the brackets denote expressions independent of L, representing the determinants of the minors after the exponential factors have been taken out of them.

We note that in each sum $\sum K_m^\alpha$ in (1.8), for each α the number n^α_+ of terms K_m^α which correspond to waves propagating to the right is equal to the number n^α_- of K_m^α which correspond to waves propagating to the left. This is a consequence of the fact that if any column corresponding to the arriving wave is used in a minor which matches the boundary condition when $x = X^\alpha$ then this column should not be used in the minors corresponding to the boundary conditions when $x = X^{\alpha-1}$ and $X = X^{\alpha+1}$.

To simplify further argument we note that for a given system of boundary conditions, using numbers α and m it is possible to determine the number of a column in the matrix Δ, and vice versa. Therefore, below we shall write C_j and K_j instead of C_m^α and K_m^α, and assume that the new subscript (in the absence of a superscript) takes on a multiplicity of values from one to the number of the last column in the matrix Δ.

If the sum in Eq.(1.8) becomes comparable with unity, then for at least one term of this sum the inequality

$$\operatorname{Im}\sum_j K_j = 0, \quad j \in \{j_i\}, \quad i = 1, 2, \ldots, 2r \tag{1.9}$$

will hold in the limit as $L \to \infty$

The set $\{j_i\}$ with respect to which the summation in (1.9) is performed is such that for each segment $[X^\alpha, X^{\alpha+1}]$ the number of terms K_j corresponding to waves which propagate to the right equals that corresponding to waves propagating to the left.

Equation (1.9) corresponds to a certain curve in the ω plane. We shall assume that as $\operatorname{Im}\omega$ decreases, a curve which corresponds to one non-zero term in the sum (1.8) will be first met in a certain range of $\operatorname{Re}\omega$, i.e. only one sum of (1.9) will vanish, and the similar sums corresponding to other terms of (1.8) will remain negative. We shall use this assumption below, and refer to it as the assumption of non-multiplicity of curve (1.9). Then it is easy to analyse the arrangement of natural frequencies in the complex plane ω. Retaining only the main term in (1.8), we can write the equation for the natural frequencies,

$$|\Delta| \approx 1 - a(\omega)\exp\sum iLK_j(\omega) = 0 \tag{1.10}$$

where $a(\omega)$ is the whole pre-exponential factor. Taking the increment of the exponent, which corresponds to $\omega - \omega_0$ where ω_0 satisfies Eq.(1.9), and assuming that $\omega - \omega_0$ is small, we obtain

$$1 - b(\omega_0)\, a(\omega_0)\exp i\,[Lc(\omega - \omega_0)] = 0 \tag{1.11}$$

$$b = \exp i\sum LK_j(\omega_0), \quad |b| = 1, \quad c = \left[\frac{\partial}{\partial\omega}\sum K_j(\omega)\right]_{\omega=\omega_0}$$

The quantity $a(\omega)$ cannot be expanded in a series since its derivative with respect to ω is finite (it does not contain L). It is obvious from (1.11) that the natural frequencies are situated at distances of the order of $1/L$ from one another, and from the line (1.9).

In the case where several lines (1.9) corresponding to several terms in (1.8) coincide, we may also assert that the natural frequencies will be situated at distances of the order of $1/L$ in the vicinity of the merged lines. If any curves (1.9), corresponding to a non-zero term in (1.8), enter the top half-plane of ω, we can choose the uppermost among them (they are all in the domain $\operatorname{Im}\omega < M$); in the neighbourhood of this curve there will be natural frequencies which cause an increase in the disturbances, i.e. instability. Instability of this type is similar to global instability (see /1/), and we refer to this as global also. It is connected with the amplification of waves corresponding to K_m^α which occur in Eq.(1.9), during their propagation, reflection and refraction.

2. If the assumption of the non-multiplicity of curve (1.9) made in Section 1 is valid, then in the sum on the right-hand side of Eq.(1.8) one term becomes of the order of unity when $\operatorname{Im}\omega$ is reduced by the first term, and the remaining terms are much less than unity, or are exactly equal to zero because the pre-exponential factor is zero. It appears in this case that the eigenfunction corresponding to the natural frequencies determined from (1.9), (1.11) presents one cyclic chain of $2r$ waves which change from one to another when there are reflections and refractions. The condition for forming this chain is that the product of the reflection and refraction coefficients corresponding to this chain, and the exponential factors determining the space changes of the amplitudes of these waves, should equal unity.

All elements in columns of the determinant of Δ which do not possess an exponent with indices of K_j in the sum (1.9) can be replaced by zeros without changing the main terms. Then, because A_ν are unit matrices we can, without altering the determinant, cross out these columns, and also the rows with the same numbers. The remaining matrix D_{ij} will have $2r$ columns and $2r$ rows, on the principal diagonal there being unit minors A_ν' remaining from A_ν. The dimension of each minor A_ν' equals the number of waves which propagate in the segment $[X^\nu, X^{\nu+1}]$ in each direction. Outside the minors A_ν' there are elements of the form

$$D_{ji} = -d_{ij}\exp iLK_j, \; i, j = 1, 2, \ldots, 2r \tag{2.1}$$

where d_{ij} is the conversion coefficient of the i-th wave into the j-th when there is interaction with the corresponding boundary. Clearly, the main terms in Eq.(1.8) can be written in the form

$$|\Delta| = 1 - |D_{ij}| = 1 + \sum(-1)^\varkappa \prod_{k,j} d_{kj}\exp i\sum_j LK_j \tag{2.2}$$

($\varkappa$ is an integer). Let us consider the non-zero term of the sum on the right-hand side of (2.2), and perform a certain rearrangement of the factors d_{ij} constituting the sequencies of factors of the form $d_{mn}\, d_{ns} \ldots d_{pq}$, so that the first subscript of each factor is identical with the second of the preceding factor. Such a group of factors, starting with the first element, is constructed uniquely and is terminated, that is, it cannot be continued when the second subscript of the last factor becomes identical with the first subscript of the first factor. We shall refer to such a chain of factors as a full cycle if it contains all $2r$ factors, and a subcycle if we have less than $2r$ factors. In the latter case we can choose one more element and build a subcycle which corresponds to it, and so on until the corresponding term in the sum (2.2) breaks up into the product of subcycles.

Cyclic chains of waves with numbers identical with the subscripts of a cycle correspond to each cycle or subcycle. The cycle itself is the product of the coefficients of mutual conversion of these waves on reflection and refraction.

It turns out that under the condition of non-multiplicity of curve (1.8), from among the terms under the symbol of the sum in (2.2) there is only one term different from zero; it corresponds to the chain of waves which are transformed into one another on reflection or refraction, at the point $x = X^{\alpha}$. The term different from zero equals the product of the mutual conversion coefficients of the waves into the coefficients of their space amplification, presented by the exponents.

To prove the above assertion we assume that all cycles whose length is less than $2r$ equal zero (because some of d_{ij} are zeros), or they are the coefficients of the exponential factors with negative indices, $\operatorname{Im} \sum LK_l < 0$ (the assumption of induction). We shall show that in this case it is impossible to form more than one cycle for a given set consisting of $2r$ waves such that $\operatorname{Im} \sum K_j = 0$. At the same time the presence of such a cycle for $\operatorname{Im} \omega > 0$ causes instability, and the absence of this cycle leads to all terms under the summation sign in (2.2) being zero, i.e. to the absence of instability connected with the chosen group of waves.

First we shall show the latter. If no full cycle exists, all the products break down into subcycles, and the sum of the exponents into groups of terms. Since the sum of all exponents is zero, at least one of these groups should have a non-negative sum. Therefore, by the assumption of induction, the corresponding cycle equals zero, and therefore all terms under the summation sign in (2.2) equal zero.

Now we shall show that if the assumption of induction is valid, no more than one full non-zero cycle can exist. Suppose we have two different cycles. We change the order of wave numbering so that the first cycle corresponds to the wave sequence $1 \to 2 \to \ldots \to 2r \to 1$, and so on. Then the second cycle will correspond to a wave sequence which can be represented by a system of arrows corresponding to this sequence. For example, the sequence shown in Fig.2 is possible for $r = 3$. The dashed line shows the sequence which corresponds to the first cycle.

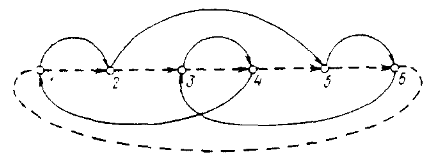

Fig. 2

Since all d_{kj} which correspond to both cycles should be non-zero, using these d_{kj} we can form subcycles different from zero. These subcycles correspond to the cyclic sequences which can be formed using the arrows corresponding to both cycles.

We can establish a correspondence between each arrow of the second cycle which does not coincide with the arrow of the first cycle, and the subcycle. For example, the subcycle consisting of four waves 2, 5, 6, 1, corresponds to the arrow $2 \to 5$. All transitions with the exception of $2 \to 5$, are taken from the first chain given. Thus, the waves whose numbers are situated under the arrow $3 \to 4$ do not enter the subcycle which corresponds to an arrow pointing to the right. Exactly as above, we can establish a correspondence between an arrow directed to the left and the cycle of waves whose numbers lie above the arrow including the ends.

Now we shall show that if we sum the partial sums of the indices $S_i = \sum K_l$ corresponding to all subcycles which can be formed by the method discussed, we obtain a quantity equal to a certain integer n multiplied by the complete system of all $2r$ indices, which by (1.9) equals zero,

$$\sum_i S_i = n \sum_{j=1}^{2r} K_j = 0 \tag{2.3}$$

To prove this assertion we note that in Fig. 2 exactly one arrow, which corresponds to the second cycle, fits each wave number. Therefore we can replace in Eq.(2.3), without upsetting it, the sums S_i corresponding to the subcycles by the sums S_i' of all indices of the subcycle with the exception of the index which corresponds to the end of an arrow.

We note also that, as the sum of all $2r$ indices is zero, the sum S_i', which corresponds to a certain arrow indicating to the right in Fig. 2 is equal, with opposite sign, to the sum of indices of numbers lying under the arrow including the arrow's end (but without the index corresponding to the initial point of the arrow). From this we can infer that the partial sum $\Sigma S_i'$ which corresponds to the continuous sequence of arrows with the first one starting at 1, consists of the terms K_i taken with the opposite sign, and corresponding to the numbers lying not more to the right than the end of the last arrow without the term K_1. Consequently, the complete sum of all S_i', and therefore of all S_i appears to be zero. For this reason at least one partial sum of S_i is non-negative.

It follows from the assumption of induction that the subcycle which corresponds to a partial non-negative sum of S_i should be equal to zero. Consequently, if at least one element d_{ij} which enters one complete cycle is zero, this same cycle itself becomes zero.

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